The Inverse Function Theorem

Christian Shadis
David MacDonald

The Inverse Function Theorem (IFT)

Let $f : (a,b) \longrightarrow \mathbb{R}$ be a continuously differentiable function, $x_0 \in (a,b)$ is a point where $f'(x_0) \neq 0$. Then there exists an open interval $I \subset (a,b)$ with $x_0 \in I$, the restriction $f|_I$ is injective with a continuously differentiable inverse $g : J \longrightarrow I$ defined on an interval $J := f(I)$, and

$$g'(y) = \frac{1}{f'(g(y))} \text{ for all } y \in J$$

Simply, if the derivative of *f* at a point is continuous and nonzero, then the function is invertible in some neighborhood around that point.

The derivative of the inverse function is given by the formula for g’(y).

Prerequisites for the Proof of IFT

- Proposition 3.2.7
- Exercise 3.2.11
- Proposition 3.6.3
- Proposition 3.6.6
- Proposition 4.2.8
- Lemma 4.4.1

Proposition 3.2.7

Let $A, B \subset \mathbb{R}$ and $f : B \to \mathbb{R}$ and $g : A \to B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g : A \to \mathbb{R}$ is continuous at c.

Proof. Let $\{x_n\}$ be a sequence in A such that $\lim x_n = c$. As g is continuous at c, then $\{g(x_n)\}$ converges to $g(c)$. As f is continuous at $g(c)$, then $\{f(g(x_n))\}$ converges to $f(g(c))$. Thus $f \circ g$ is continuous at c. □

If a function is continuous at a point (g continuous at c)**, and a second function f** (mapped from g's codomain) **is continuous at g(c), then the composition of the two functions is continuous at c.**

Intuitively, if we input a continuous function's image to another continuous function (<u>composition</u>), then the result is continuous.

Exercise 3.2.11

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Suppose $f(c) > 0$. Show that there exists an $\alpha > 0$ such that for all $x \in (c - \alpha, c + \alpha)$ we have $f(x) > 0$.

Main idea: If a function at a point (f(c)) is positive, then there exists a small neighborhood around that point for which f(x) for all x in the neighborhood is also positive.

Proof of Exercise 3.2.11

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Suppose $f(c) > 0$. Show that there exists an $\alpha > 0$ such that for all $x \in (c - \alpha, c + \alpha)$ we have $f(x) > 0$.

Proof. Choose ϵ such that $f(c) - \epsilon > 0$. This is possible by the Archimedean Property. Then there exists some $\delta > 0$ such that if $x \in \mathbb{R}$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$ by the definition of continuity. Therefore
$|f(x) - f(c)| < \epsilon \ \forall x \in (c - \delta, c + \delta)$.
$\implies -\epsilon < f(x) - f(c) < \epsilon \ \forall x \in (c - \delta, c + \delta)$.
$\implies f(c) - \epsilon < f(x) < \epsilon + f(c) \ \forall x \in (c - \delta, c + \delta)$.
Since $f(c) - \epsilon > 0 \implies 0 < f(x) \ \forall x \in (c - \delta, c + \delta)$.
Let $\alpha = \delta$. Thus we have found $\alpha > 0$ such that $\forall x \in (c - \alpha, c + \alpha)$, $f(x) > 0$.

$\square$

Corollary 3.6.3

If $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is monotone and not constant, then $f(I)$ is an interval if and only if f is continuous.

Similar to the intermediate value theorem:

- If *f* is continuous across an interval, then its image will contain every value in an interval.

Main ideas of proof (proof long and tedious):

- Suppose *f* continuous
  - Take two points (x1, x2)
  - Use Intermediate Value Theorem to find an arbitrary point between them to find an arbitrary point in *f(I)*, and thus it is an interval (x1, x2)
- Suppose *f(I)* is an interval - prove by contrapositive
  - Introduce a discontinuity
  - Show that there are point(s) missing in the potential interval *f(I)*.

Proposition 3.6.6

If $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is strictly monotone, then the inverse $f^{-1} : f(I) \to I$ is continuous.

Main ideas of proof (proof long and tedious):

- If *f* is strictly increasing or decreasing, then so is its inverse in the same manner.
- Consider the left and right limits as the inverse approaches an arbitrary point
  - They can be written as the supremum and infimum of subsets of *I*.
  - The left and right limits are the same because *f* is strictly monotone.

Proposition 4.2.8

Let I be an interval and let $f : I \to \mathbb{R}$ be a differentiable function. Then
(i) If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing.
(ii) If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing.

Main ideas of Proof:

Part (i):

1. Consider f'(x) > 0 for all x in I = (a, b)
2. Choose two values (x, y) in I such that a < x < y < b
    a. Use Mean Value Theorem to show that f(y) > f(x) and thus f is strictly increasing

Part (ii):

1. Consider f'(x) < 0 for all x in I = (a, b)
2. Choose two values (x, y) in I such that a < x < y < b
    a. Use Mean Value Theorem to show that f(y) < f(x) and thus f is strictly decreasing

Lemma 4.4.1

Let $I, J \subset \mathbb{R}$ be intervals. If $f : I \longrightarrow J$ is strictly monotone (hence one-to-one), onto ($f(I) = J$), differentiable at $x_0 \in I$, and $f'(x_0) \neq 0$, then the inverse f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}.$$

If f is continuously differentiable and f' is never zero, then f^{-1} is continuously differentiable.

Proof of Lemma 4.4.1

Proof. By Proposition 3.6.6, f has a continuous inverse. Name the inverse $g : J \longrightarrow I$. Let x_0, y_0 be defined as in the statement. For any $x \in I$, $y := f(x)$. If $x \neq x0$ and so $y \neq y0$,

$$\frac{g(y)-g(y_0)}{y-y_0} = \frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)} = \frac{x-x_0}{f(x)-f(x_0)}.$$

Let $Q(x) := \begin{cases} \frac{x-x_0}{f(x)-f(x_0)}, & x \neq x_0 \\ \frac{1}{f'(x_0)}, & x = x_0 \end{cases}$

Since f is differentiable at x_0,

$$\lim_{x\to x_0} Q(x) = \lim_{x\to x_0} \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{f'(x_0)} = Q(x_0)$$

so Q is continuous at x_0. As $g(y)$ is continuous at y_0, the composition $Q(g(y)) = \frac{g(y)-g(y_0)}{y-y_0}$ is continuous at y_0 by Proposition 3.2.7. Therefore

$$\frac{1}{f'(g(y_0))} = Q(g(y_0)) = \lim_{y\to y_0} Q(g(y)) = \lim_{y\to y_0} \frac{g(y)-g(y_0)}{y-y_0}.$$

So g is differentiable at y_0 and $g'(y_0) = \frac{1}{f'(g(y_0))}$. If f is continuous and nonzero at all $x \in I$, then the lemma applies at all $x \in I$. As g is also continuous (since it is differentiable), the derivative $g'(y) = \frac{1}{f'(g(y))}$ must be continuous.

$\square$

Combining what we know...

Proof of Inverse Function Theorem

Let $f : (a, b) \longrightarrow \mathbb{R}$ be a continuously differentiable function, $x_0 \in (a, b)$ is a point where $f'(x_0) \neq 0$. Then there exists an open interval $I \subset (a, b)$ with $x_0 \in I$, the restriction $f|_I$ is injective with a continuously differentiable inverse $g : J \longrightarrow I$ defined on an interval $J := f(I)$, and

$$g'(y) = \frac{1}{f'(g(y))} \text{ for all } y \in J$$

Proof. Without loss of generality, suppose $f'(x_0) > 0$. As f' is continuous, there must exist an open interval $I = (x_0 - \delta, x_0 + \delta)$ such that $f'(x) > 0$ for all $x \in I$ by the conclusion in Exercise 3.2.11. By Proposition 4.2.8, f is then strictly increasing on I, and hence the restriction $f|_I$ is bijective onto $J := f(I)$. As f is continuous, then by the Corollary 3.6.3 $f(I)$ is an interval. Then by Lemma 4.4.1, $f^{-1} := g$ is differentiable at $y_0 = f(x_0)$, and $g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}$.

□

Application of the Theorem

Let $f : (-\infty, \infty) \to \mathbb{R}$ be defined by $f(x) = e^x$. Notice f is continuously differentiable as $f'(x) = e^x$. Also notice $f'(x) \neq 0$ for all $x \in \mathbb{R}$. By the Inverse Function Theorem, for all $x_0 \in \mathbb{R}$, there exists an $I \subset \mathbb{R}, x_0 \in I$, the restriction $f|_I$ is injective with a continuously differentiable inverse $g : f(I) \to I$ and, for all $y \in J$,

$$g'(y) = \frac{1}{f'(g(y))}$$

Since we have intervals for all $x \in \mathbb{R}$, it follows that this applies for f on all of $\mathbb{R}$. Then,

$$g'(y) = \frac{1}{f'(g(y))}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{e^{\ln x}}$$

$$\frac{d}{dx}(\ln x) = {}^1/_x$$

References

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