1.12B

Prove that for all $n \geq 2$, $n^3 - n$ is divisible by 6.
Proof by Mathematical Induction:
Base Case: Let $n = 2$: $2^3 - 2 = 6$, and $6|6$.

Inductive Hypothesis: For all n up to and including $n = k$, let $6|k^3 - k$.
Expand the expression $k^3 - k$ to $k \cdot (k+1) \cdot (k-1)$
By 1.12A, we know that $k^2 + k$, which is equivalent to $k \cdot (k+1)$, is even.
Thus, $k^2 + k = 2l$ for some $l \in \mathbb{Z}$ by the definition of an even number.

Inductive step: now consider the case of $n = k + 1$
$n^3 - n$ becomes $(k+1)^3 - (k-1)$
$(k+1)^3 - (k-1) = k^3 + 2k^2 + k + k^2 + 2k + 1 - k - 1$
After combining like terms and grouping, the expression can be rewritten as the following:
$(k^3 - k) + (3k^2 + 3k)$
$3k^2 + 3k = 3 \cdot (k(k+1))$
By the inductive hypothesis, we know that $k(k+1) = 2l$ for some $l \in \mathbb{Z}$
So, $(3k^2 + 3k)$ can be written as $6l$ for some $l \in \mathbb{Z}$
Thus, $(k^3 - k) + (3k^2 + 3k)$ can be rewritten as $(k^3 - k) + 6l$ for some $l \in \mathbb{Z}$
By the Inductive Hypothesis, we also know that $6|(k^3 - k)$
By the definition of divisibility, this means $k^3 - k = 6m$ for some $m \in \mathbb{Z}$
Thus, $(k^3 - k) + 6l$ can be rewritten as $6m + 6l$ for some l,$m \in \mathbb{Z}$
Factoring out the 6 yields the expression $6 \cdot (m + l)$ for some l,$m \in \mathbb{Z}$

Since $(m + l) \in \mathbb{Z}$, $n^3 - n$ is divisible by 6 for all $n \in \mathbb{N}$ by the principle of mathematical induction.

1.23A

Prove $a + c \equiv b + d \pmod{m}$.
Let $a \equiv b \pmod{m}$
Let $c \equiv d \pmod{m}$
Let a, b, c, d, $m \in \mathbb{Z}$ with $m > 0$
By 1.22, since $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, $m|a - b$ and $m|c - d$
So: $a - b = mx$ for some $x \in \mathbb{Z}$
and $c - d = my$ for some $y \in \mathbb{Z}$
Thus, $a = b + mx$ and $c = d + my$ for some x,$y \in \mathbb{Z}$
Adding $a + c$ yields the following expression:
$a + c = b + mx + d + my$
Subtracting $(b + d)$ from both sides yields the following expression:
$(a + c) - (b + d) = mx + my$
This can be rewritten as:
$(a + c) - (b + d) = m(x + y)$
Since $x + y \in \mathbb{Z}$, $m|(a + c) - (b + d)$
Since $m|(a + c) - (b + d)$, we know that $a + c \equiv b + d \pmod{m}$ by 1.22.

If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$